Cumulative Damage and Fatigue Life Prediction

Tiiu V. Kutt*
AT&T Bell Laboratories, Whippany, New Jersey
and
M. P. Bieniek†
Columbia University, New York, New York

A cumulative damage rule is proposed for fatigue of metals under variable stress-amplitude loading. The rule is nonlinear and takes into account the sequence of stress levels, i.e., high-to-low or low-to-high changes of stress amplitudes. To facilitate probabilistic estimates of safety of structural elements subjected to fatigue loading, a stochastic model of fatigue damage is developed. The mean value and the variance of the fatigue life of an element are determined in terms of the statistics of the material properties and of the load parameters.

Nomenclature

b	= parameter describing *S-N* diagram ($\tan^{-1}b$ is the inverse slope of a linearized version of log-log plot of the *S-N* curve)
c	= exponent of constant stress amplitude damage function
c_i	= exponent of constant stress amplitude damage function for stress amplitude S_i
D	= constant stress amplitude damage function
D_i	= fatigue damage after application of ith stress block
ΔD_i	= damage increment due to application of ith stress amplitude
$\partial D/\partial x$	= rate of change of damage with respect to cycle ratio
$E(\ )$	= expected value
f	= stress-interaction function
k_{ji}	= defined by Eq. (7)
m	= material parameter used in the stress interaction function
n_i	= number of applied cycles at stress amplitude S_i
N_i	= number of cycles to failure at constant stress amplitude S_i
N_0	= endurance limit of *S-N* curve
P_f	= probability of failure
S_i	= ith stress amplitude
Var( )	= variance
V_c, V_D, V_n, V_N	= coefficients of variation of $c,D,n,$ and N, respectively
x_i	= cycle ratio, $\equiv n_i/N_i$
$x_{i,i-1}$	= equivalent cycle ratio to produce damage D_{i-1} at stress amplitude S_i
y	= exponent describing damage curves
z_i	$\equiv 1/N_i$
η	= defined by Eq. (19)
$\mu_c, \mu_D, \mu_n, \mu_N$	= mean values of $c,D,n,$ and N, respectively

Introduction

THE major problems in the evaluation of the fatigue strength of a structural element appear to be the following:

1) The formulation of a simple and reliable cumulative damage rule for loadings with variable stress amplitudes.

2) The formulation of a rational stochastic model of cumulative damage that will correctly reflect the randomness of the material properties and the stochastic character of external loads present in most structures.

3) The choice of a suitable measure of stress intensity at a point of the structure, i.e., the choice between the nominal stress vs the actual peak stress in complex shapes (notches), the inclusion of the effect of residual stresses due to the manufacturing process or to the past load history, and the question of what the "effective stress" is in the case of two- or three-dimensional states of stress.

4) Occasionally, in the presence of complicated load histories, the question of what constitutes a "cycle of loading" and in what way the load cycles should be counted in order to arrive at a "load spectrum" that will indeed be equivalent to the actual stress-time history at a point of the structure.

The earliest, and still very often used, method of fatigue analysis is based on the Palmgren-Miner rule of cumulative damage.[1,2] This rule is probably the simplest possible; it utilizes the absolute minimum amount of material data and, with proper care, is capable of providing the structural analyst with useful information concerning the severity of the fatigue environment of a structure. Nevertheless, the method has failed to satisfy the increasing demand for accuracy and reliability of the fatigue life predictions of various structures. Also, none of the subsequent developments and improvements of the phenomenological, or stress-type, fatigue theories have gained significant acceptance among structural analysts. In fact, the trend in the past decade has been away from the stress-type fatigue theories and toward fracture mechanics approaches. Among the latter, the Paris formula[3] has gained substantial recognition and widespread use.

There is no doubt that the fracture mechanics (crack growth) analyses are to some extent free from the faults of the early stress-type analyses. Most important, they are *nonlinear* in determining the cumulative damage, which is a feature badly missing in the Palmgren-Miner theory. They also address to some degree the problem of the proper stress measure in the structural element by dealing with the stress intensity at the crack tip rather than the continuum-type stress state. (This last advantage, however, is often obtained at the cost of prohibitively complicated and still not fully reliable analyses of stress fields in cracked elements, e.g., by the finite-element method with variable meshes.) In certain aspects, however, the fracture mechanics theories are as inadequate as the stress-type theories. The problem, for example, of the crack growth rate under variable-amplitude cy-

Presented as Paper 85-0614 at the AIAA/ASME/ASCE/AHS 26th Structures, Structural Dynamics and Materials Conference, Orlando, FL, April 15–17, 1985; received April 1, 1986; revision received June 26, 1987.

*Member of Technical Staff.

†Professor, Department of Civil Engineering and Engineering Mechanics. Member AIAA.

cling is as little understood as the classical fatigue damage accumulation under these conditions.

The authors believe that a rational stress-type fatigue theory may provide a practical and reliable tool of analysis and design of structures under cyclic loadings. Such a theory need not represent a "counterpoint" to the fracture mechanics methods. In fact, a truly rational stress-type theory should be consistent with, and properly reflect, the microscopic processes in solids; its predictions should be in good agreement with the predictions of any other rational (e.g., fracture mechanics) theory.

This work represents an attempt at the compilation of a stress-type theory of cumulative fatigue damage and fatigue life prediction. Among the four major problems of the fatigue theory mentioned earlier in this section, this work deals with the first two, namely, the cumulative damage rule and the stochastic model of fatigue damage.

Background

Cumulative Damage

The experimental data on the fatigue failure of specimens subjected to cyclic loads, as well as the data on the crack growth rate under cyclic stresses, have led to certain widely accepted notions concerning the nature of the fatigue damage. The most fundamental of them are:

1) The fatigue damage, say D, in constant-stress cycling is a nonlinear function of the number of applied stress cycles. Moreover, the form of this function depends on the magnitude of the stress amplitude S. This last property of the damage is referred to as "stress dependency" and the corresponding theories are said to be "stress dependent." The nonlinearity and stress dependency of the fatigue damage function are illustrated in Fig. 1 in "normalized" variables $x \equiv n/N$ (where n is the number of applied cycles at stress amplitude S and N the number of cycles to failure at the same stress amplitude) and D such that $D=1$ at failure.

2) The incremental fatigue damage ΔD_i caused by a "block" of n_i cycles at some stress amplitude S_i depends on the magnitude of damage D_{i-1}, due to the previous cycling history of blocks of $n_1, n_2, \ldots, n_{i-1}$, and on the relation of the previous stress amplitudes $S_1, S_2, \ldots, S_{i-1}$ to the stress amplitude S_i. The effect of the previous damage D_{i-1} on the incremental damage ΔD_i is already included in the assumption of the nonlinearity of the damage function. The effect of the magnitude of the stress amplitudes $S_1, S_2, \ldots, S_{i-1}$ in relation to S_i (i.e., the "low-high" vs "high-low" sequence of stress amplitudes) is independent of the properties stated in item 1 above; it is referred to as the "stress-interaction" effect of the fatigue damage. The stress-interaction effect has been clearly established in numerous fatigue strength and crack propagation tests. Although its quantitative formulation appears to be somewhat elusive, the following aspects seem to be well recognized:

a) A block of low-stress cycles reduces the magnitude of the rate of damage caused by the subsequent block of high-stress cycles; in other words, strain aging at low stresses increases the fatigue resistance in subsequent high-stress cycling.

b) A block of high-stress cycling (in a smooth specimen) *increases* the rate of damage in subsequent low-stress cycling.

c) A single (or only a few) high-stress peak of overload *reduces* the rate of damage in subsequent cycling at stress amplitudes lower than the peak stress. The same phenomenon has been observed in the rate of crack growth.

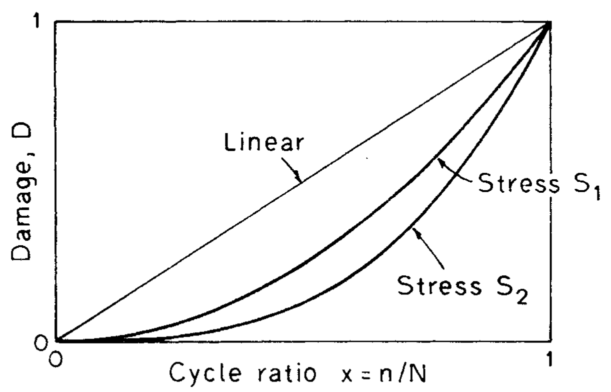

Fig. 1 Linear and nonlinear stress-dependent damage functions.

Several important works that followed the 1945 paper by Miner[2] contained cumulative damage theories incorporating the nonlinearity of the damage rate and, in some instances, the stress-interaction effects. It should be noted here that nonlinear but stress-independent cumulative damage rules fail to yield results in agreement with the available experimental evidence. The assumption, for example, of the damage due to n cycles of constant-amplitude stress in the form

$$D = (n/N)^c \tag{1}$$

(where c is a constant) leads to the following damage due to a block of n_1 cycles at S_1 followed by a block of n_2 cyles at S_2:

$$D = [(n_1/N_1) + (n_2/N_2)]^c \tag{2}$$

With failure defined as $D=1$, the failure criterion in terms of the cycle ratios becomes

$$(n_1/N_1) + (n_2/N_2) = 1$$

i.e., the linear relation of the Miner rule, which is in disagreement with experimental evidence.

Nonlinear, stress-dependent, interaction-free theories were proposed in the 1950's by Shanley[3] and Marco and Starkey.[5] Corten and Dolan[6] developed a nonlinear, stress-independent, interaction theory that, in some respects, reflects the microscopic changes in the material. The theory by Freudenthal and Heller[7] introduced the interaction and nonlinear effects in terms of an S-N relation that was adjusted empirically to the load spectrum characteristics. The theory by Grover,[8] the "double linear damage rule" by Manson et al.,[9] and the "double exponential rule by Miller and Zachariah[10] made use (although qualitatively only) of the fact that the fatigue damage of the material progresses in two stages: crack initiation and crack propagation. These considerations led to nonlinear, stress-dependent, interaction-free damage rules. A different nonlinear, stress-dependent, interaction-free damage rule was proposed by Marin[11] on the basis of the notion of certain "isodamage curves." The isodamage curves are, essentially, S-N diagrams that are adjusted to account for the damage caused by past cyclic history. Extensive use of the "damage curve" and "residual life" concepts was made in the theory developed by Hashin and Rotem.[12] It contained a nonlinear, stress-dependent, interaction-free rule; its important feature was that all the necessary material data can be obtained from the S-N diagram resulting from constant-amplitude fatigue tests.

Nonlinear, stress-dependent, interaction-free theories were proposed in the 1950's by Shanley[4] and Marco and Starkey.[5] Corten and Dolan[6] developed a nonlinear, stress-independent, interaction theory that, in some respects, reflects the microscopic changes in the material. The theory by Freudenthal and Heller[7] introduced the interaction and nonlinear effects in terms of an S-N relation that was adjusted empirically to the load spectrum characteristics. The theory by Grover,[8] the "double linear damage rule" by Manson et al.,[9] and the "double exponential rule" by Miller and Zachariah[10] made use (although qualitatively only) of the fact that the fatigue damage of the material progresses in two stages: crack initiation and crack propagation. These considerations led to nonlinear, stress-dependent, interaction-free damage rules. A different nonlinear, stress-dependent, interaction-free damage rule was proposed by Marin[11] on the basis of the notion of certain "isodamage curves." The isodamage curves are, essentially, *S-N* diagrams that are adjusted to account for the damage caused by past cyclic history. Extensive use of the "damage curve" and "residual life" concepts was made in the theory developed by Hashin and Rotem.[12] It contained a nonlinear, stress-dependent, interaction-free rule; its important feature was that all the necessary material data can be obtained from the *S-N* diagram resulting from constant-amplitude fatigue tests.

Stochastic Models

The large scatter of experimental data in the tests of fatigue strength of materials has led to the general acceptance of the notion that the progress of fatigue damage is essentially a random process. Consequently, the methods of the theory of probability and statistics have become indispensable in the quantitative description of various aspects of fatigue of materials. The works in this area can be divided into three groups:

1) The first group deals with the description of the fatigue life of a material subjected to constant-amplitude cycling. The fatigue life, expressed in terms of the number of cycles to failure, is a random variable and a substantial effort has been devoted to the problem of accurate determination of its probability function. A survey of the pertinent literature can be found in Ref. 13.

2) With the development of the theory of random vibrations in mechanical systems and structures, research effort has been focused on the problem of fatigue life of an element of a machine or a structure subjected to random loadings, in which the stress history is a random process. The emphasis of this work has been placed on the randomness of the loading, with the models of the fatigue cumulative damage remaining deterministic and extremely simple in fact, employing simply the Palmgren-Miner rule. The works by Miles,[14] Powell,[15] Crandall et al.,[16] Bendat,[17] and Wirsching and Light[18] are the most prominent examples of research on fatigue due to random loadings.

3) The third group of works on the problem of stochastic modeling of fatigue deals with the cumulative fatigue damage as a random process. The rationale of this research is that the randomness of the cumulative fatigue damage is due to the random nature of the material response to stress cycles and to the randomness of the stress cycles themselves in structural and machine elements under random loads. The results of this research appear to have direct application to the problem of fatigue life prediction and reliability of structures and machine components. The work that influenced the present research was done by Parzen,[19] Bogdanoff,[20] and Hashin.[21]

Parzen's model of fatigue failure is based on the linear cumulative damage rule and certain assumptions concerning the probabilistic nature of damage under individual cycles of stress. They lead to the notion of modeling the fatigue damage as a *renewal process.* Using some results of the renewal theory, the mean value and the variance of the damage function D can be determined and, thus, the probability of $D \geq 1$, i.e., the probability of failure can be estimated. Bogdanoff's assumptions concerning the cumulative damage caused by a "duty cycle" of an element result in a *Markoff process* as the model of fatigue damage accumulation in the element. The probabilistic theory proposed by Hashin is based on the Hashin-Rotem rule of cumulative damage,[12] with the quantities determining the residual life of an element assumed to be random variables. Consequently, the residual life itself is a random variable whose mean value and variance can be determined. A stochastic model for the related problem of fatigue crack propagation has been proposed by Lin and Yang.[22] In their model, the crack growth rate is described by a *random pulse train* multiplied by a function of the (current) crack size, the stress intensity factor and its range, the stress, and the stress ratio.

Proposed Cumulative Damage Rule

The basic aspects of the proposed cumulative damage rule satisfy the following requirements:

1) The damage function D, under constant-amplitude cycling, depends on the cycle ratio $x \equiv n/N$ and the stress amplitude S, with D being zero at the start of loading and equal to 1 at failure, i.e.,

$$D = D(x,s)$$

$$\text{with } D(0,S) = 0 \text{ and } D(1,S) = 1$$

2) The function $D(x,S)$ increases monotonically with x, with its rate $\partial D/\partial x$ also increasing with x.

3) After a change in stress amplitude from, say, S_{i-1} to S_i, the rate of damage at the new stress level depends on the existing magnitude of damage, on the stress level S_i, and on the stress change $(S_{i-1} - S_i)$. The last effect is expressible in the form

$$\frac{\partial D}{\partial x} = f\,(\text{stress history}) \cdot \left(\frac{\partial D}{\partial x}\right)^*$$

where $(\partial D/\partial x)^*$ is the damage rate without the effect of the stress step $(S_{i-1} - S_i)$ and f the interaction function whose purpose is to introduce quantitatively the effect of the sequence the high- and low-stress amplitudes (i.e., the high-low vs low-high sequence of cycles).

4) Limiting the scope of the present theory to the case of moderately sized blocks of cycles at different stress amplitudes, the following conditions are assumed for the interaction function f:

$$f > 1 \text{ if } S_{i-1} > S_i$$

$$f < 1 \text{ if } S_{i-1} < S_i$$

with $|f - 1|$ increasing as $|S_{i-1} - S_i|$ increases.

The present assumptions concerning the function f take into account only two effects, namely, 1) acceleration of the damage rate if the current stress level is preceeded by a higher stress level, and 2) reduction of the damage rate if the current stress level is preceded by a lower stress level. The favorable effect of a single overload and a more precise effect of the number of cycles in the consecutive stress blocks is, at present, omitted from the proposed rule.

The proposed damage function for constant-amplitude cycling has the form

$$D = x^c \tag{4}$$

where $c = c(S)$ is a function of stress $[c(S) > 1]$ to be determined experimentally.

The interaction function f is assumed as

$$f = 1 + m\left(\frac{S_1 - S_2}{S_2}\right)x_1 \tag{5a}$$

for two-stage loading (two stress levels) at amplitudes S_1 and S_2 and with the number of cycles n_1 and n_2, respectively; $x_1 \equiv n_1/N_1$ and, in general,

$$f_i = 1 + m \sum_j \left(\frac{S_j - S_i}{S_i}\right)x_j \tag{5b}$$

where m is a material parameter.

The dependence of the damage increments ΔD on the existing damage magnitude, which is implied by the nonlinearity of Eq. (4), leads to the following generalization of the damage function at the end of the ith block of stress cycles (at stress S_i and with n_i cycles):

$$D_i = D_{i-1} + f_i[(D_{i-1}^{1/c_i} + x_i)^{c_i} - D_{i-1}] \qquad (6)$$

where $c_i \equiv c(S_i)$. The term D_{i-1}^{1/c_i} in Eq. (6) is sometimes called the "equivalent cycle ratio" at the stress level S_i relative to the damage D_{i-1} and denoted by $x_{i,i-1}$.

The case of two-stage loading serves as the test of the ability of the above rule to represent the experimentally established fatigue behavior of a material, to determine the material parameters $c(S)$ and m, and to compare the present damage rule with some of the rules listed in the preceding section. For two stress levels S_1 and S_2, with the number of cycles n_1 and n_2, respectively, and with $x_1 \equiv n_1/N_2$ and $x_1 \equiv n_2/N_2$, the present rule [Eq. (6)] results in the magnitude of damage at the end of the second stage

$$D_2 = x_j^{c_1} + (1 + k_{12}x_1)[(x_1^{c_1/c_2} + x_2)^{c_1} - x_1^{c_2}] \qquad (7)$$

where

$$k_{12} = m\frac{S_1 - S_2}{S_2}$$

With the failure criterion $D_2 = 1$, Eq. (7) yields the following relation between x_2 and x_1:

$$x_2 = \left(\frac{1 + k_{12}x_1^{c_1+1}}{1 + k_{12}x_1}\right)^{1/c_2} - x_1^{c_1/c_2} \qquad (8a)$$

For the same test case, the Marco-Starkey rule becomes

$$x_2 = 1 - x_1^{c_1/c_2} \qquad (8b)$$

and the Marin rule becomes

$$x_2 = (S_1/S_2)^{y-b}(1 - x_1) \qquad (9)$$

where b and y are the inverse slope of the S-N curve and an exponent to be determined by experiment, respectively. The Hashin-Rotem rule

$$x_2 = 1 - x_1^{\mathrm{Log}(N_2/N_0)/\mathrm{Log}(N_1/N_0)} \qquad (10)$$

where N_0 is a material parameter (the endurance limit in the S-N curve). Finally, note that Eq. (3) follows from the Miner rule or a nonlinear but stress-independent and interaction-free rule.

The material parameters of the present theory, $c(S)$ and m, have been determined from the two-level tests by Manson et al.[8] The results of the failure predictions with the aid of the present cumulative damage rule [Eq. (8)] and with the Palmgren-Miner rule are shown in Figs. 2 and 3 for the high-low and low-high loading sequences, respectively. The material data from the constant-amplitude tests and the parameters used in the present cumulative damage rule are listed in Table 1.

The stress dependency of the damage function through the exponent $c(S)$ and the presence of the stress interaction function $f(S_1 - S_2)$ have similar effect on the shape of the x_2 vs x_1 curve at failure by bringing it below the Palmgren-Miner line in the case of high-low load sequences and above the Palmgren-Miner line for low-high load sequences. It must be noted, however, that neither the stress dependency nor the interaction alone is capable of accurate predictions of failure points. For example, the strengthening effect of the initial low-stress cycling visible in Figs. 3b-3d cannot be reproduced by the stress dependency alone.

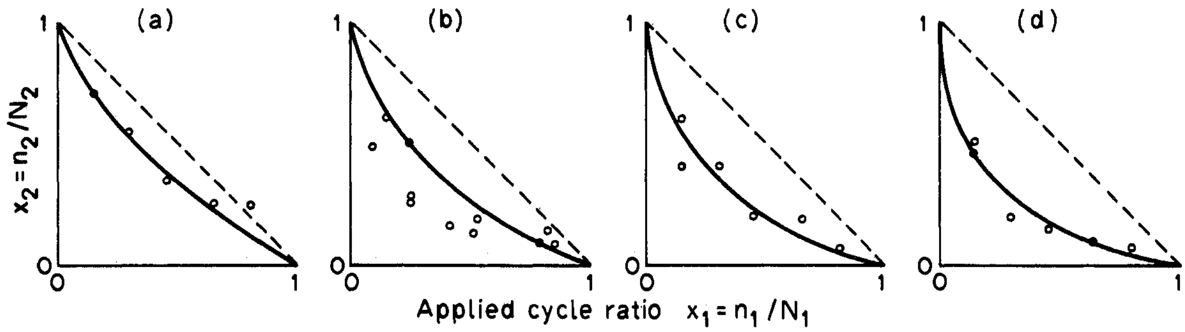

Fig. 2 Four cases of high-low, two-level cycling: a) S_1 = 290 ksi, S_2 = 240 ksi; b) S_1 = 240 ksi, S_2 = 140 ksi; c) S_1 = 290 ksi, S_2 = 160 ksi; d) S_1 = 290 ksi, S_2 = 120 ksi (solid line represents the present rule, broken line the Palmgren-Miner rule, dots the experiment from Ref. 8).

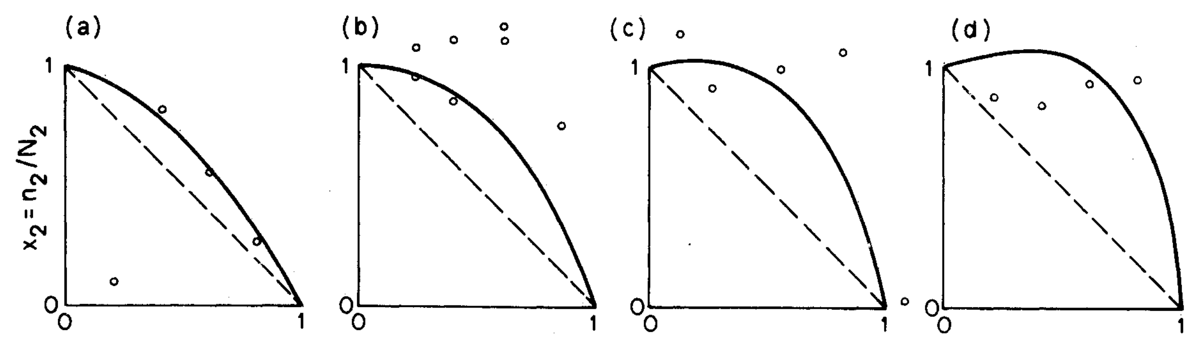

Fig. 3 Four cases of low-high, two-level cycling: a) S_1 = 240 ksi, S_2 = 290 ksi; b) S_1 = 140 ksi, S_2 = 240 ksi; c) S_1 = 160 ksi, S_2 = 290 ksi; d) S_1 = 120 ksi, S_2 = 290 ksi (solid line represents the present rule, broken line the Palmgren-Miner rule, dots the experiments from Ref. 8).

Table 1 Material data for the two-level cycling shown in Figs. 2 and 3

Stress amplitude S_i ksi	120	140	160	240	290
Cycles to failure N_i from constant-amplitude tests	244,000	94,000	44,000	3800	1280
Exponent of the nonlinear damage function $c(S_i)$	5.9	5.6	4.8	3.0	2.0
Parameter m of the interaction function	1.0	1.0	1.0	1.0	1.0

Stochastic Model of Fatigue Damage

The present approach to the formulation of a stochastic model of fatigue damage is based on the assumption that the damage D is a function of the random variables describing the material properties and stress history. Specifically, the damage D_i at the end of n_i cycles of stress S_i given by Eq. (6) depends on the material variables $N_1,N_2,\ldots,N_i$ and $c_1,c_2,\ldots,c_i$ and the loading variables $n_1,n_2,\ldots,n_i$ and $S_1,S_2,\ldots,S_i$. Adoption of this assumption implies that the random damage increment ΔD_i due to the stress block S_i of n_i cycles can be attributed deterministically to the individual cycles within the block. This is, of course, a different physical statement than that forming the basis of the renewal theory approach to the damage function D, namely, that the fractional damages due to individual cycles within a block are random variables with identical probability functions.

A complete description of the damage D_i requires specification of its probability distribution function $F_D(x_i)$. The function F_D, in turn, determines the probability of failure, which is the probability that $D_i \geq 1$, for a given x_i, i.e., for a given service life of the element being analyzed. It appears, however, that the full description of the probability function F_D is, at the present time, not possible. There are two main reasons for this situation: 1) the exact distribution functions of the random variables that are the arguments of D_i are not known, and 2) the mathematical difficulties involved in the full determination of F_D are considerable. Consequently, following the previous efforts in this area and the general trend of the methods of the probabilistic theory of structural reliability, the objective of the present stochastic model is the determination of only the first two moments of F_D, namely, the mean value $E(D_i)$ and variance $\mathrm{Var}(D_i)$ of damage D_i.

In order to provide an illustration of the Palmgren-Miner rule, the damage function is

$$D_i = \sum_{j=1}^{j=i} \frac{n_j}{N_j} \tag{11}$$

With D treated as a function of n_j and N_j, its mean value and variance are

$$E(D) = \sum_j E(n_j) E\left(\frac{1}{N_j}\right) \tag{12}$$

$$\mathrm{Var}(D) = E\left\{\left[\sum_j \left(\frac{n_j}{N_j}\right)\right]^2\right\} - [E(D)]^2 \tag{13}$$

Approximate expressions for $E(D)$ and $\mathrm{Var}(D)$ can be obtained by using approximations for $E(1/N_j)$ and $\mathrm{Var}(1/N_j)$, which yield

$$E(D) \cong \sum_j \frac{E(n_j)}{E(N_j)}(1+V_{Nj}^2) \tag{14}$$

$$\mathrm{Var}(D) \cong \sum_j \left[\frac{E(n_j)}{E(N_j)}\right]^2 \left[V_{Nj}^2(1-V_{Nj}^2)(1+V_{nj}^2) + V_{nj}^2(1+V_{Nj}^2)^2\right] \tag{15}$$

where V_{nj} and V_{Nj} are the coefficients of variation of n_j and N_j, respectively.

By contrast, if the function D given by Eq. (11) is treated as a renewal process, the asymptotic theorems of the renewal theory yield the expression for $E(D)$ and $\mathrm{Var}(D)$ in the form

$$E(D) = \sum_j \frac{E(n_j)}{E(N_j)} \tag{16}$$

$$\mathrm{Var}(D) = \sum_j \frac{E(n_j)\,\mathrm{Var}(N_j)}{E^3(N_j)} + \mathrm{Var}\left[\sum_j \frac{n_j}{E(N_j)}\right] \tag{17}$$

The above three methods of computation of $\mathrm{Var}(D)$ have been applied to a specific example of variable-amplitude cycling of specimens of 7075-T6 aluminum alloy reported in Ref. 23. The results of these calculations, as well as the experimental histograms of fatigue life, are shown in Fig. 4. In all of the calculations, the distribution function of D is assumed to be log-normal; consequently,

$$P(D \geq 1) = \Phi(-\eta) \equiv \frac{1}{\sqrt{2\pi}} \int_\eta^\infty e^{-y^2/2} dy \tag{18}$$

where

$$\eta = \frac{-[\ln(\mu_D) - \ln(\sqrt{1+V_D^2})]}{\sqrt{\ln(1+V_D^2)}} \tag{19}$$

The stochastic model based on the cumulative damage rule proposed in the preceding section can, in principle, reflect the contribution of all the random aspects of the material properties and loading. However, due to the lack of sufficient experimental data on some of the material parameters [notably, the exponents $c(S_i)$] in the damage function given by Eq. (4) and in order to achieve a possibly simple theory, the following probability models are assumed for the material properties and stress history:

1) The number of cycles to failure $N(S_j)$ at a constant stress amplitude S_j is a random variable. In the present theory, the description of $N(S_j)$ in terms of its mean value and variance are used. Both of these quantities can be estimated from a series of constant-amplitude fatigue tests.

2) The stress history is described in terms of a typical "duty cycle" block of stress amplitudes $S_1,S_2,\ldots,S_j,\ldots,S_i$, with each stress level occurring $n(S_i)$ times. The numbers of stress cycles $n(S_j)$ is assumed to be random variable with a mean and a variance.

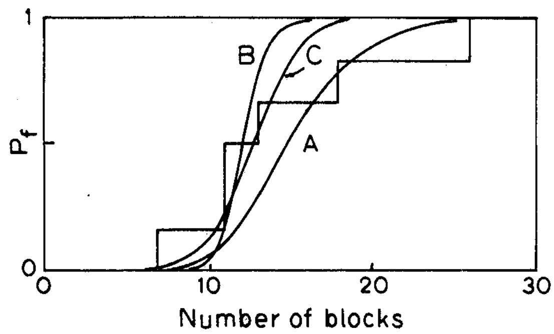

Fig. 4 Predictions of the probability of failure based on Miner's rule. Curve A: Eqs. (12) and (13); curve B: Eqs. (14) and (15); curve C: using renewal theory of Eqs. (16) and (17).

Consider first the effect of the randomness of $N(S_j) \equiv N_j$. Since the damage function D depends on N_j in a somewhat complicated way, an approximate form of $D(N_j)$ is obtained first by expanding $D(N_j)$ in the Taylor series about the mean of $N_j \equiv \mu_{Nj}$, i.e.,

$$D = D(\mu_N) + \sum_j (N_j - \mu_{Nj}) \frac{\partial D}{\partial N_j}\bigg|_{\mu_N} + \frac{1}{2!}\left[\sum_j (N_j - \mu_{Nj})^2 \frac{\partial^2 D}{\partial N_j^2}\bigg|_{\mu_N} + \sum_i \sum_j (N_i - \mu_{Ni})(N_j - \mu_{Nj}) \frac{\partial^2 D}{\partial N_i \partial N_j}\bigg|_{\mu_N}\right] + \ldots \quad (20)$$

and truncating the series after the fourth term. In the case of N_j being statistically independent, the mean and the variance of D then follow as

$$E(D) \cong D(\mu_N) + \frac{1}{2!}\sum_j \frac{\partial^2 D}{\partial N_j^2}\bigg|_{\mu_N} \mathrm{Var}(N_j) + \frac{1}{3!}\sum_j \frac{\partial^3 D}{\partial N_j^3}\bigg|_{\mu_N} E[(N_j - \mu_{Nj})^3] \quad (21)$$

and

$$\mathrm{Var}(D) = E(D^2) - E^2(D) \quad (22)$$

Somewhat simpler calculations are needed if $1/N_j$ rather than N_j is used as the independent variable of the function D. The Taylor expansion of D is then

$$D = D(\mu_z) + \sum_j (z_j - \mu_{zj}) \frac{\partial D}{\partial z_j}\bigg|_{\mu_z} + \frac{1}{2!}\left[\sum_j (z_j - \mu_{zj})^2 \frac{\partial^2 D}{\partial z_j^2}\bigg|_{\mu_z} + \sum_i \sum_j (z_i - \mu_{zi})(z_j - \mu_{zj}) \frac{\partial^2 D}{\partial z_i \partial z_j}\bigg|_{\mu_z}\right] + \ldots \quad (23)$$

which leads to $E(D)$ in the form

$$E(D) \cong D(\mu_z) + \frac{1}{2!}\sum_j \frac{\partial^2 D}{\partial z_j^2}\bigg|_{\mu_z} \mathrm{Var}(z_j) + \frac{1}{3!}\sum_j \frac{\partial^3 D}{\partial z_j^3}\bigg|_{\mu_z} E[(z_j - \mu_{zj})^3] \quad (24)$$

where $z_j \equiv 1/N_j$ and the corresponding expression for $\mathrm{Var}(D)$ is obtained according to Eq. (22).

The mean value of $1/N_j$ and the variance of $1/N_j$ can be estimated from the data obtained in constant-amplitude tests. Alternately, these values can be computed from the mean and the variance of N_j; approximate formulas for this purpose are

$$\mu_{1/N} \cong \frac{1}{\mu_N}(1 + V_N^2) \quad (25)$$

$$V_{1/N}^2 \cong \frac{V_N^2(1 - V_N^2)}{(1 + V_N^2)^2} \quad (26)$$

Equations (23-26) have been applied to the case of the test program reported in Ref. 23, in which spectrum loading was applied to specimens of aluminum alloy 7075-T6. The stress sequence in a typical "duty cycle" block is shown in Fig. 5. Table 2 contains the number of cycles of each stress amplitude as well as the material data used in these calculations. The numbers of cycles to failure have been taken from constant-amplitude tests of Ref. 23.

The results of the present predictions of the probability of failure and the actual histograms of fatigue life are shown in Fig. 6. Although the case of mixed high-low and low-high stress sequences is relatively favorable to the stress-independent, interaction-free rules, it can be noticed that the present predictions are closer to the experimental data than any of the results shown in Fig. 4. Specifically, the present theory predicts more conservative fatigue lives than those in Fig. 4; also, it yields a greater scatter of the fatigue lives. The diagrams of $E(D)$ and $\mathrm{Var}(D)$ as functions of the load cycles (or numbers of load blocks) are shown in Fig. 7.

The randomness of the applied stress cycles $n(S_j) \equiv n_j$ can be taken into account in a similar manner. First, the truncated Taylor expansion of D in n_j and N_j is performed; then, the mean value and the variance of D are determined resulting in the following equation of $E(D)$:

Table 2 Load and material data for spectrum loading of Al 7055-T6

S_i, ksi	3.8	9.1	15.0	21.2	27.2	33.7	39.9	46.3
n (per block)	24,400	4800	690	98	14	1.8	0.33	0.074
c	7.7	5.2	3.7	2.7	2.0	1.5	1.3	1.2

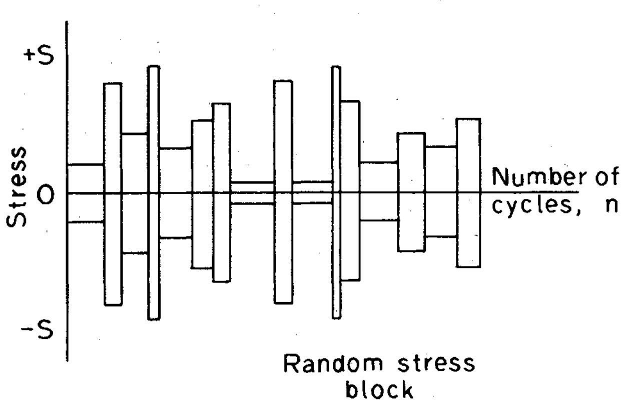

Fig. 5 Stress sequence in typical "duty cycle" blocks (see Ref. 23).

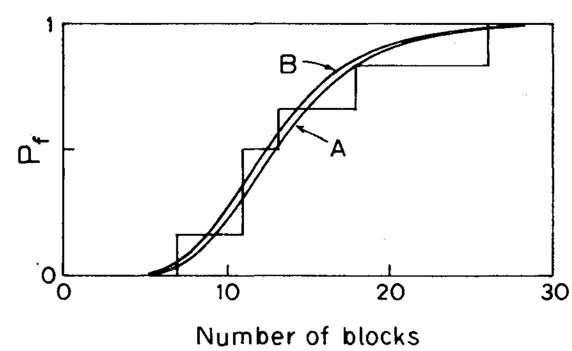

Fig. 6 Predictions of the probability of failure based on the present theory. Curve A: deterministic load cycles, random fatigue life under constant stress; curve B: random load cycles and random fatigue life under constant stress; experimental data from Ref. 23.

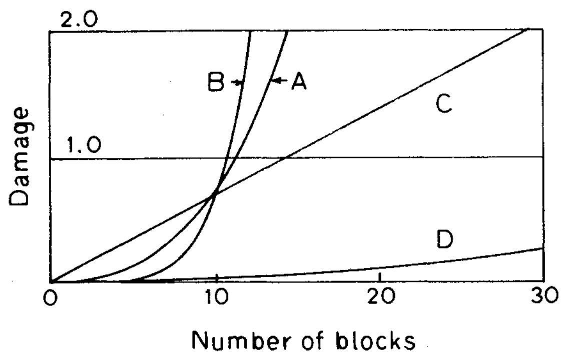

Fig. 7 Mean value and variance of damage—curves A and B: according to the present theory [Eqs. (24) and (22)]; curves C and D: according to the Palmgren-Miner rule [Eqs. (12) and (13)].

$$E(D) \cong D(\mu_n, \mu_z) + \frac{1}{2!} \sum_j \frac{\partial^2 D}{\partial z_j^2}\bigg|_{\mu_n,\mu_z} \mathrm{Var}(z_j) + \frac{1}{2!} \sum_j \frac{\partial^2 D}{\partial n_j^2}\bigg|_{\mu_n,\mu_z} \mathrm{Var}(n_j) + \frac{1}{3!} \sum_j \frac{\partial^3 D}{\partial z_j^3}\bigg|_{\mu_n,\mu_z} E[(z_j - \mu_{zj})^3] + \frac{1}{3!} \sum_j \frac{\partial^3 D}{\partial n_j^3}\bigg|_{\mu_n,\mu_z} E[(n_j - \mu_{nj})^3] \tag{27}$$

and an equation for Var(D) expressed by Eq. (22).

Conclusions

The main features of the proposed cumulative damage rule are nonlinearity, stress dependency, and stress interaction. It appears that all these features are necessary to achieve a satisfactory agreement between the theoretical predictions of cycles to failure and the experimental data from fatigue tests under variable amplitude loads, especially under two-stage (high-low and low-high) cycling.

The stochastic model, based on the present damage rule, is capable of taking into account the randomness of the material strength (reflected in the probability distribution of the cycles to failure at a given stress level) as well as the randomness of the cycles of applied loads. The resulting predictions of fatigue life of an element are in the form of the probability of failure as a function of the expected (mean) number of cycles of applied loads. The probability of failure is by itself a meaningful quantitative measure of the strength of an element; it can be also used in arriving at rational values of the conventional safety factors, paralleling the ideas of Ref. 24.

Further research is needed in clarification of the role of the true local stresses as opposed to the nominal or average stresses in an element. Closely related to this is the problem of residual stresses caused by either the manufacturing processes or the past history of loading. A more refined stress interaction function must be capable of discerning both the favorable and unfavorable effects of past overloadings or low-stress aging. Finally, the phenomenological (or "strength") theories, the various microscopic considerations (crystalline slips, microstresses), and the fracture mechanics methods must be brought to a much better mutual consistency than what exists today.

Acknowledgment

This paper is based on the doctoral dissertation of the first author submitted to the Department of Civil Engineering and Engineering Mechanics at Columbia University.[25] It contains more detailed derivations and discussion of the subject of this paper.

References

[1] Palmgren, A.Z., "Die Lebensdauer von Kugellagern," *Zeitschrift des Vereines der Deutschen Ingenioeren,* Vol. 68, 1924, pp. 339-341.

[2] Miner, M.A., "Cumulative Damage in Fatigue," *Journal of Applied Mechanics,* Vol. 12, 1945, pp. A159-A164.

[3] Paris, P.C., "The Fracture Mechanics Approach to Fatigue," *Fatigue—An Interdisciplinary Approach, Proceedings of the 10th Sagamore Army Material Research Conference,* Syracuse University Press, Syracuse, NY, 1964, pp. 107-132.

[4] Shanley, F.R., "Theory of Fatigue Based on Unbonding During Reversed Slip," RAND Corp., Nov. 1952, p. 350.

[5] Marco, S.M. and Starkey, W.L., "A Concept of Fatigue Damage," *Journal of Applied Mechanics,* Vol. 76, 1954, pp. 627-643.

[6] Corten, H.T. and Dolan, T.J., "Cumulative Fatigue Damage," *Proceedings of the International Conference on Fatigue of Metals,* IME and ASME, New York, 1956, pp. 235-246.

[7] Freudenthal, A.M. and Heller, R.A., "On Stress Interaction in Fatigue and a Cumulative Damage Rule," *Journal of the Aerospace Sciences,* Vol. 26, No. 7, 1959, pp. 431-442.

[8] Grover, H.J., "An Observation Concerning the Cycle Ratio in Cumulative Damage," *Symposium of Fatigue of Aircraft Structures,* STP-274, ASTM, Philadelphia, 1960, pp. 120-124.

[9] Manson, S.S., Freche, J.C. and Ensign, C.R., "Application of a Double Linear Damage Rule to Cumulative Fatigue," *Fatigue Crack Propagation,* STP-415, ASTM, Philadelphia, 1966, pp. 384-412.

[10] Miller, K.J. and Zachariah, K.P., "Cumulative Damage Laws for Fatigue Crack Initiation and Stage I Propagation," *Journal of Strain Analysis and Engineering Design,* Vol. 12, 1977, pp. 262-270.

[11] Marin, J., "Simple Stress Fatigue Properties—Varying Amplitudes," *Mechanical Behavior of Engineering Materials,* Prentice-Hall, Englewood Cliffs, NJ, 1962, pp. 197-204.

[12] Hashin, Z. and Rotem, A., "A Cumulative Damage Theory of Fatigue Failure," *Materials Science and Engineering,* Vol. 34, 1978, pp. 147-160.

[13] Freudenthal, A.M. and Gumbel, E.J., "Physical and Statistical Aspects of Fatigue," *Advances in Applied Mechanics,* Vol. 4, Academic Press, New York, 1956, pp. 117-158.

[14] Miles, J.W., "On Structural Failure under Random Loading," *Journal of the Aeronautical Sciences,* Vol. 21, 1954, pp. 753-762.

[15] Powell, A.W., "On the Fatigue Failure of Structures Due to Vibrations Excited by Random Pressure Fields," *Journal of the Acoustical Society of America,* Vol. 30, 1958, pp. 1130-1135.

[16] Crandall, S.H., Mark, W.D., and Khabbaz, G.R., "The Variance in Palmgren-Miner Damage Due to Random Vibration," *Proceedings of 4th U.S. National Congress on Applied Mechanics,* Vol. 1, 1962, pp. 119-126.

[17] Bendat, J.S., "Probability Functions for Random Responses: Prediction of Peaks, Fatigue Damage, and Catastropic Failures," NASA CR-33, 1964.

[18] Wirsching, P.H. and Light, M.C., "Fatigue Under Wide Bank Random Stresses," *ASCE Journal of the Structural Division,* Vol. 106, No. ST 7, 1980, pp. 1593-1607.

[19] Parzen, E., "On Models for the Probability of Fatigue Failure of a Structure," AGARD Rept. 245, 1959.

[20] Bogdanoff, J.L., "A New Cumulative Damage Model," *Journal of Applied Mechanics,* Vol. 45, 1978, Pt. I, pp. 246-250; Pt. II, pp. 251-257; Pt. III, pp. 733-739.

[21] Hashin, Z., "Statistical Cumulative Damage Theory for Fatigue Life Prediction," *Journal of Applied Mechanics,* Vol. 50, 1983, pp. 571-579.

[22] Lin, Y.K. and Yang, J.N., "A Stochastic Theory of Fatigue Crack Propagation," *AIAA Journal,* Vol. 23, Jan. 1985, pp. 117-124.

[23] Naumann, E.C., Hardrath, H.F., and Guthrie, D.E., "Axial-Load Fatigue Tests of 2024-T3 and 7075-T6 Aluminium Alloy Sheet Specimens Under Constant- and Variable-Amplitude Loads," NASA TN D-212, 1959.

[24] Bieniek, M.P. and Joanides, J.C., "Safety Factors and the Probability of Failure in Fatigue," *AIAA Journal,* Vol. 9, April 1971, pp. 753-754.

[25] Kutt, T.V., "Cumulative Damage and Fatigue Life Prediction," Ph.D. Dissertation, Dept. of Civil Engineering and Engineering Mechanics, Columbia University, New York, 1985.